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Boundary-layer separation at a free streamline. Part 1. Two-dimensional flow

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The boundary-layer flow just upstream of the trailing edge of a flat plate is studied when a free streamline is attached to the edge. The separation at the edge occurs with an infinitely favourable pressure gradient and is characterized by a skin friction which is proportional to the inverse eighth power of the distance from the edge. The proportionality factor for the first-order term is independent of the upstream boundary-layer flow. The streamwise velocity profile at separation is non-analytic near the wall $Y = 0$, and starts with the term $Y^{\frac{2}{3}}$.

1. Introduction

This paper is a study of the boundary-layer separation that occurs at the sharp trailing edge of a flat plate when there is a free streamline attached to the edge. Problems of this type occur in a number of physical situations, the most notable cases arising from cavitation and hydraulic applications involving free streamline flows. The important feature which distinguishes this separation from the usual boundary-layer separation is the sign and magnitude of the pressure gradient imposed on the boundary layer by the potential flow. In classical separation the pressure gradient is adverse while here the gradient is extremely favourable, with a singularity right at the edge. It is of practical and theoretical interest to find a description of these flows near the singularity.

In the past many authors have failed to distinguish any basic difference between cavitation and boundary-layer separation (see, for example, Lighthill's comments in Rosenhead (1963, p. 4)). This confusion arose, no doubt, from the early use of the free streamline theory in the analytical treatment of both phenomena. Except for this dubious similarity, these phenomena occur in totally different physical situations. Although free streamlines are an approximation to flows with separation, they accurately describe cavitating flows in which a *favourable* pressure gradient is the prelude to the cavitation.

In some ways free streamline separation is easier to study because the point of separation may be well-defined by the potential flow. In addition, the boundary layer remains thin up to the separation point and cannot induce 'breakaway', the situation in an adverse pressure gradient wherein a very thick boundary layer causes the potential flow to leave the wall. The validity of the boundary-layer approximation has been questioned for such cases with good reason, but for the flows discussed here, the boundary-layer approximation should be quite accurate to within a very small distance upstream of the free streamline. To avoid any confusion, the boundary-layer phenomena considered here will be called 'shearaway' because the separation point is one of infinite shear, as we shall see.†

The analytical study of boundary-layer separation in an adverse pressure gradient was first considered by Goldstein (1948) in his classic paper. Although this work left some minor questions unanswered, later to be resolved by Stewartson (1958), it was correct in all essential details and very few new ideas have been introduced since. For more current views on separation, the interested reader is referred to Brown & Stewartson (1969) and Kaplun (1967). In this paper our method is similar to Goldstein's, the main difference being the form of the terminal velocity profile, which must be deduced in the course of the analysis.

In §2, the problem of 'shearaway' is formulated mathematically. The form of the potential flow solution and the pressure gradient near the free streamline are derived in §3. An 'inner' boundary-layer solution of similarity form is sought in §4. The resulting differential equation involves a balance of inertia, pressure, and viscous terms, and the domain of validity of its solution is expected to be larger than if fewer terms had been retained. Nevertheless, the boundary condition at the edge of the boundary layer cannot be satisfied and to obtain a uniformly valid solution, an 'outer' expansion is found in §5 which merges with the similarity solution via the matching technique. The principal result indicates that the skin friction just upstream of the free streamline is singular, to first order, and proportional to the inverse eighth power of the distance from the edge. The proportionality constant depends on the potential flow solution and is independent of the upstream boundary-layer motion. The upstream conditions enter the skin-friction expansion at the second order by means of an eigenfunction. The streamwise velocity profile at the point of 'shearaway' starts with the power $Y^{\frac{3}{4}}$, where Y is a measure of the distance normal to the wall.

In §6, a comparison of the theory with some finite difference calculations is made and the agreement is quite good. Finally, in §7 the results are summarized and discussed.

2. Boundary-layer equations

We introduce a co-ordinate system with the origin located at the point of separation and the negative $\bar{x}$ axis directed upstream along the plate (see figure 1). The co-ordinate $\bar{y}$ is the distance normal to the plate measured positively into the fluid. Denoting dimensional variables by bars, we introduce the following non-dimensional variables:

$$x = \bar{x}/L, \quad Y = \bar{y}R^{\frac{1}{2}}/L, \quad u = \bar{u}/U_0, \quad v = \bar{v}R^{\frac{1}{2}}/U_0, \quad p = \bar{p}/\rho U_0^2. \tag{2.1}$$

Here L is a length scale characteristic of the assumed potential flow outside the boundary layer, U_0 is the fluid speed along the free streamline, ρ is the constant fluid density and $R = \rho U_0 L/\mu$ is the Reynolds number, μ being the viscosity.

† The names 'boundary-layer separation' and 'breakaway' have been associated with points of zero shear in the boundary-layer flow.

When $R \to \infty$, Prandtl's boundary-layer equations are applicable† and may be written in the form

$$\partial u/\partial x + \partial v/\partial Y = 0 \quad (x \leqslant 0,\ Y \geqslant 0), \tag{2.2}$$

$$u\,\partial u/\partial x + v\,\partial u/\partial Y = U_e dU_e/dx + \partial^2 u/\partial Y^2 \quad (x \leqslant 0,\ Y \geqslant 0), \tag{2.3}$$

where $U_e(x)$ is the non-dimensional x component of velocity just outside the boundary layer and is determined from the potential-flow solution.

The boundary conditions require

$$u(x,0) = 0 = v(x,0) \quad \text{for} \quad x < 0, \tag{2.4}$$

$$u(x,Y) \to U_e(x) \quad \text{for} \quad x < 0, \quad Y \to \infty. \tag{2.5}$$

Equations of the type (2.2) and (2.3) also require an initial velocity profile to specify a solution downstream. However, for flows involving separation, the

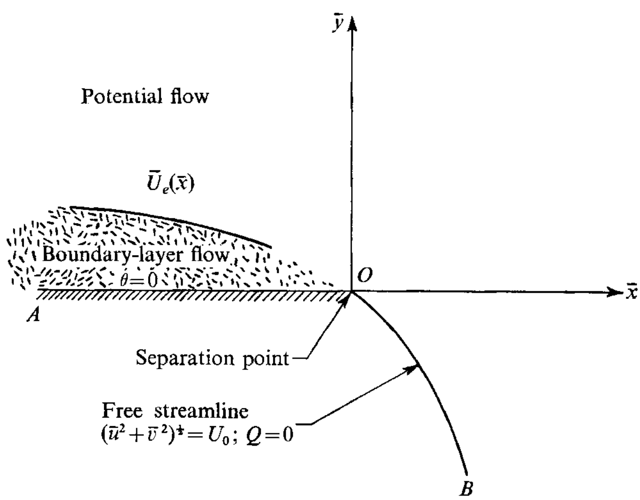

FIGURE 1. Sketch of geometry and $\bar{Z}$ plane.

problem is formulated differently. In these cases a separation (or terminal) velocity profile $U_s(Y)$ is assumed or deduced at the separation point ($x = 0$) and a description of the flow field is required upstream. Although an initial profile may be quite arbitrary, a terminal profile will usually be restricted because the solution it implies upstream must satisfy the boundary conditions and be free of singularities.‡ These points have been discussed in detail by Stewartson (1957, p. 174) and Kaplun (1967, p. 254 ff.) in a physical way; much remains to be done in clarifying these ideas with rigor. The main point is that $U_s(Y)$ will not usually be

† The boundary-layer equations fail in a small neighbourhood [$x = o(1)$] of a trailing edge as Stewartson (1968) has pointed out. An analysis similar to Stewartson's (1969) 'triple deck' could be used to correct the boundary-layer solution in this region. The solution studied here will be correct for $x = O(1)$.

‡ Stated in another way, there is no reason to expect that an arbitrary terminal profile could have evolved from any physically relevant initial profile, with specified boundary conditions, and with a solution in between which is free of singularities.

known in advance, although one might guess the first few terms as did Goldstein (1948). Here the form of $U_s(Y)$ cannot easily be anticipated. Thus, we simply write

$$u(0, Y) = U_s(Y) \quad \text{for} \quad Y \geqslant 0, \tag{2.6}$$

and the form of U_s will be deduced in the course of the analysis.

Our problem then is to solve (2.2) and (2.3) subject to the boundary conditions (2.4), (2.5), and terminal condition (2.6) in $x \leqslant 0$, $Y \geqslant 0$.

3. The potential flow near the point of separation

Before (2.2) and (2.3) can be solved, the potential flow solution must be used to determine the forcing term $U_e dU_e/dx$. When boundary-layer separation occurs in an adverse pressure gradient, the validity of this procedure is questionable because the thick separated layer may alter the assumed potential flow. This difficulty does not arise here because (1) the position of separation is fixed in the potential-flow solution and (2) it will be shown that the boundary layer separates with an infinitely favourable pressure gradient and thus remains thin up to the separation point.

The nature of a potential flow near smooth and abrupt separation points has been discussed by Armstrong (1953) and Thwaites (1960, p. 165). A brief analysis will be given here to consider features of the flow which are important for the boundary-layer motion.

We introduce the complex variable $Z = x+iy$, and define the complex velocity potential $W(Z) = \Phi + i\Psi$ and complex velocity $dW/dZ = U - iV$. Each function will be analytic in Z (excepting isolated singular points) and thus the logarithm of the non-dimensional complex velocity

$$\Gamma(W) = Q(\Phi, \Psi) - i\theta(\Phi, \Psi) = \ln(dW/dZ), \tag{3.1}$$

will be an analytic function of W. The fluid deflexion θ is related to the logarithm of the non-dimensional speed, Q, by the Cauchy–Riemann equations. Here Z and W have been non-dimensionalized with respect to L and LU_0, respectively. We choose $W = 0$ at the point of separation, and the traces of the Z plane in the W and Γ planes are shown in figures 2 and 3.

The potential solution near point O can be found by locally mapping the Γ plane onto the W plane. This transformation yields an explicit differential equation of the form (3.1). Applying the Schwarz–Christoffel transformation we find

$$d\Gamma/dW = k' W^{-\frac{1}{2}} G(W), \tag{3.2}$$

where $G(W)$ is an arbitrary function, analytic in the neighbourhood of $W = 0$, with a Taylor series expansion

$$G(W) = 1 + \sum_{n=1}^{\infty} \alpha_n W^n, \tag{3.3}$$

and k' and α_n are constants. Upon substituting (3.3) into (3.2), integrating term-by-term and requiring $\Gamma = 0$ when $W = 0$, we find

$$\Gamma(W) = k'[2W^{\frac{1}{2}} + (\tfrac{2}{3})\alpha_1 W^{\frac{3}{2}} + (\tfrac{2}{5})\alpha_2 W^{\frac{5}{2}} + \ldots]. \tag{3.4}$$

Putting $W = \rho e^{i\gamma}$ with $0 \leqslant \gamma \leqslant \pi$ and noting,

$$Q = 0 \quad \text{for } \gamma = 0, \tag{3.5}$$

$$\theta = 0 \quad \text{for } \quad \gamma = \pi, \tag{3.6}$$

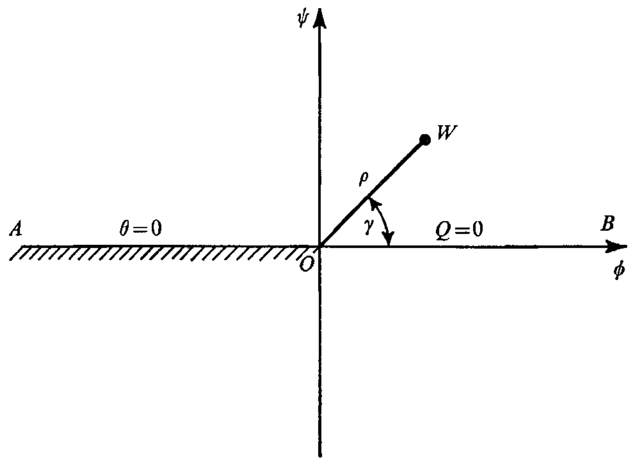

FIGURE 2. W plane.

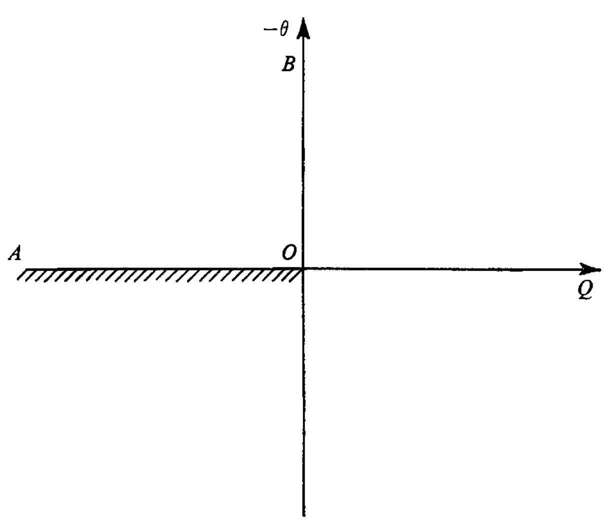

FIGURE 3. Γ plane.

the final form of (3.4) may be written

$$\Gamma(W) = -ke^{-\frac{1}{2}\pi i}\, W^{\frac{1}{2}}[2+(\tfrac{2}{3})\alpha_1 W+(\tfrac{2}{5})\alpha_2 W^2+\ldots], \tag{3.7}$$

where the constants k and α_n must be real. Substituting (3.7) in (3.1) yields a differential equation for W which may be solved subject to the condition $W(0) = 0$, i.e.

$$Z = W+\tfrac{4}{3}ke^{-\frac{1}{2}\pi i}W^{\frac{3}{2}}+k^2e^{\pi i}\, W^2+O(W^{\frac{5}{2}}). \tag{3.8}$$

Using (3.7) and (3.8) and noting that along the wall AO, $dW/dZ = U$ and $W = \rho e^{\pi i}$, we find after some algebra

$$U(x,0) = 1-2k(-x)^{\frac{1}{2}}-\tfrac{10}{3}k^2x+O[(-x)^{\frac{3}{2}}] \quad \text{for} \quad x<0. \tag{3.9}$$

Identifying $U_e(x)$ with $U(x,0)$, we finally obtain

$$-\frac{1}{\rho}\frac{dp}{dx} = U_e\frac{dU_e}{dx} = \frac{k}{(-x)^{\frac{1}{2}}}+(\tfrac{16}{3})k^2+O((-x)^{\frac{1}{2}}) \quad \text{for} \quad x<0. \tag{3.10}$$

Thus the pressure gradient is infinite for $x \to 0-$ and for the physical situations considered in this paper, the constant k will be positive. When $k<0$, the pressure gradient is infinitely adverse and the boundary layer would no doubt have separated upstream of O. It is noteworthy that the first two terms in (3.10) depend on the single constant k of the potential flow.

4. Similarity solution

For problems where $U_e(x)$ monotonically increases as the free streamline is approached it is convenient to use

$$t = 1-U_e(x) \quad (t \geqslant 0), \tag{4.1}$$

as a new independent variable in place of x (see, for example, Ackerberg & Glatt 1968).† Introduce the stream function $\Psi(t, Y)$ with

$$u = \partial\Psi/\partial Y \quad \text{and} \quad v = U_e'\partial\Psi/\partial t. \tag{4.2}$$

Here primes denote differentiation with respect to x. Equation (2.3) may be written

$$-\Psi_Y\Psi_{Yt}+\Psi_t\Psi_{YY} = 1-t+(U_e')^{-1}\Psi_{YYY}, \tag{4.3}$$

where subscripts denote partial differentiation. The boundary conditions (2.4)–(2.6) require

$$\Psi_Y = 0 = \Psi_t \quad \text{for} \quad Y=0, \quad t \geqslant 0, \tag{4.4}$$

$$\Psi_Y \to 1-t \quad \text{for} \quad Y \to \infty, \quad t \geqslant 0, \tag{4.5}$$

and

$$\Psi_Y \to U_s(Y) \quad \text{for} \quad t \to 0+, \quad Y \geqslant 0. \tag{4.6}$$

When (3.9) is used to express $(U_e')^{-1}$ in terms of t, we find

$$(U_e')^{-1} = (t/2k^2)[1+\tfrac{5}{2}t+\ldots] = (2k^2)^{-1}t\sum_{n=0}^{\infty} c_n t^n, \tag{4.7}$$

where the c_n's are constants of the potential flow and k is the constant in (3.9).

An 'inner' asymptotic solution of (4.3) is sought in the form

$$\Psi^i(t, Y) = 2^{\frac{1}{2}}k\xi^{\beta}F(\xi,\eta), \tag{4.8}$$

where

$$\eta = 2^{\frac{1}{2}}kY/t^{\alpha}, \quad \xi = t^{\alpha} \quad (\eta, \xi \geqslant 0), \tag{4.9}$$

and α and β are constants to be found. This solution is expected to be valid for $\eta = O(1)$ and $\xi \to 0$. The velocity components are given by

$$u = 2k^2\xi^{\beta-1}F_\eta, \quad v = 2^{\frac{1}{2}}k\alpha U_e'\xi^{\beta-(1/\alpha)}(\beta F-\eta F_\eta+\xi F_\xi). \tag{4.10}$$

† In many problems of physical interest the monotone requirement will be met because the boundary layer will commence at a stagnation point (attachment) and terminate at the free streamline.

When (4.7), (4.8) and (4.9) are substituted in (4.3), we obtain

$$\alpha\xi^{2\beta-2-(1/\alpha)}[\beta F F_{\eta\eta}-(\beta-1)F_\eta^2+\xi(F_\xi F_{\eta\eta}-F_\eta F_{\xi\eta})]$$
$$= 1-\xi^{1/\alpha}+[\xi^{1/\alpha}+\tfrac{5}{2}\xi^{2/\alpha}+\ldots]\xi^{\beta-3}F_{\eta\eta\eta}. \qquad (4.11)$$

To find a solution which has the largest range of validity, it is plausible to balance inertia, pressure, and viscous stress terms (as far as possible) in the differential equation. Such a balance is obtained if we choose

$$\alpha = \tfrac{3}{4}, \quad \beta = \tfrac{5}{3}, \qquad (4.12)$$

with F having the form $$F(\xi,\eta) = \sum_{n=0}^{\infty} \xi^{\frac{4}{3}n}F_n(\eta).\dagger \qquad (4.13)$$

The equations for F_0 and F_n are

$$F_0''' - \tfrac{5}{4}F_0F_0''+\tfrac{1}{2}F_0'^2+1 = 0, \qquad (4.14)$$

$$F_n''' - \tfrac{5}{4}F_0F_n''+(n+1)F_0'F_n'-(n+\tfrac{5}{4})F_0''F_n = G_n \quad \text{for} \quad n \geqslant 1, \qquad (4.15)$$

where G_n is a function of the F_m's ($m < n$) and the coefficients c_n appearing in (4.7). There may also be forcing terms due to eigenfunctions.

The no-slip condition requires

$$F_n(0) = 0 = F_n'(0) \quad \text{for} \quad n \geqslant 0, \qquad (4.16)$$

where now a prime denotes differentiation with respect to η. To satisfy (4.6) a matching condition, to be discussed in §5, will require that

$F_n(\eta)$ must not contain any exponentially large terms for $\eta \to \infty$. (A)

(a) *The function $F_0(\eta)$*

A series expansion for F_0 about the origin is given by

$$F_0(\eta) = \sum_{n=0}^{\infty} a_n\eta^{n+2}, \qquad (4.17)$$

where the a_n's are known in terms of a_0. Efforts to integrate (4.14) numerically, starting at $\eta = 0$, were not successful because a small error in a_0 yields a solution which develops a singularity of the form $3(\eta^*-\eta)^{-1}$ where $\eta^* > 0$.

The numerical solution was obtained by integrating backwards, starting with the asymptotic expansion,

$$F_0(\eta) \sim \sum_{n=0}^{\infty} A_n\eta^{\frac{1}{3}(5-n)} \quad \text{for} \quad \eta\to\infty, \qquad (4.18)$$

where A_0 and A_3 are arbitrary and

$$A_1 = A_2 = A_5 = 0, \quad A_4 = 9(5A_0)^{-1}, \quad A_6 = A_3^2(5A_0)^{-1}, \text{ etc.} \qquad (4.19)$$

Values for A_0 and A_3 were chosen and the integration proceeded backwards from $\eta = 20$ using a fourth-order Runge–Kutta method. New values for A_0 and

† For the time being, the sum in (4.13) will be restricted to integer values; however, 'eigenvalues' are not only possible but necessary for a physically sensible formulation. In some cases 'eigenfunctions' may be exponentially small (see Ackerberg 1968, p. 1287) and they would not appear in a series of this form. In physical variables (x, Y), the similarity variable $\eta \propto Y/(-x)^{\frac{3}{8}}$.

A_3 were computed using Newton's method until (4.16), with $n = 0$, was satisfied to $O(10^{-7})$. The final results for F_0 and F_0' are displayed in figure 4 and

$$A_0 = 1{\cdot}950718\ldots, \quad A_3 = -1{\cdot}577568\ldots, \quad F_0''(0) = 3{\cdot}014015\ldots. \tag{4.20}$$

(b) The functions $F_n(\eta)$

There are three independent complementary solutions of (4.15) with series expansions about $\eta = 0$ starting with multiples of

$$1, \ \eta, \ \eta^2. \tag{4.21}$$

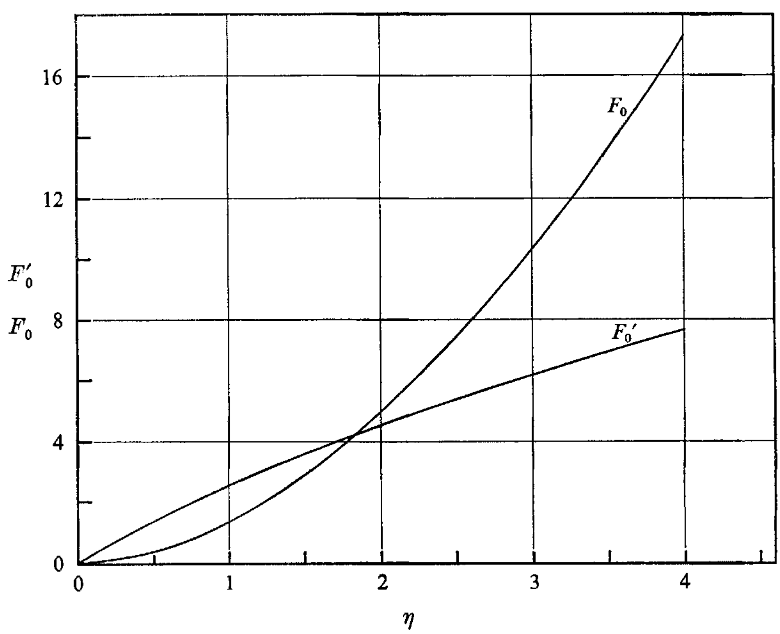

FIGURE 4. $F_0(\eta)$ and $F_0'(\eta)$ *versus* η.

For $\eta \to \infty$, three complementary solutions may be found whose asymptotic expansions start with multiples of

$$\eta^{\frac{2}{3}}, \quad \eta^{\frac{1}{3}(4n+5)}, \quad \eta^{-\frac{1}{3}(4n+8)} \exp\{\tfrac{15}{32} A_0 \eta^{\frac{8}{3}}\}. \tag{4.22}$$

Solutions for F_n may be obtained in the following way: First a particular solution F_n^p is determined which satisfies (4.16). In general, this solution will contain a multiple of the exponentially large term, shown in (4.22), in its asymptotic expansion.† To satisfy the asymptotic condition (A) it is necessary to add to F_n^p a multiple of a complementary solution satisfying (4.16) which will eliminate the exponentially large term. This procedure will always be possible unless complementary solutions exist which satisfy both (4.16) and (A); such solutions will be called eigenfunctions.

† If we use mathematical induction, the forcing terms in (4.15) will, by hypothesis, be algebraically large and will contribute only algebraically large terms to F_n^p.

(c) Eigenfunctions

To see that eigenfunctions are formally possible we first note that a complementary solution of (4.15) is any multiple of F_0'. We substitute

$$F_n = F_0' w, \tag{4.23}$$

into (4.15), neglect the right-hand side, and put $z = w'$. The resulting equation is

$$F_0' z'' + (3F_0'' - \tfrac{5}{4}F_0 F_0')z' + (3F_0''' - \tfrac{5}{2}F_0 F_0'' + \lambda F_0'^2)z = 0, \tag{4.24}$$

where $\lambda = n+1$. The boundary conditions require

$$|z(0)| < \infty \text{ with } z(0)/z'(0) = F_0''(0), \tag{4.25}$$

and $z(\eta)$ should have no exponentially large terms for $\eta \to \infty$.

Using an integrating factor, (4.24) may be written in the self-adjoint form

$$(F_0'^3 e^{-q} z')' + F_0'^2 e^{-q} P z = 0, \tag{4.26}$$

where $q(\eta) = \frac{5}{4}\int_0^\eta F_0(\eta)\,d\eta$ and P is the multiplier of z in (4.24). If (4.26) is multiplied by z, integrated from 0 to ∞, and solved for λ we find

$$\lambda = \frac{\int_0^\infty F_0'^3 e^{-q} z'^2\,d\eta - \int_0^\infty F_0'^2 e^{-q}(3F_0''' - \frac{5}{2}F_0 F_0'')z^2\,d\eta}{\int_0^\infty F_0'^4 e^{-q} z^2\,d\eta}.\dagger \tag{4.27}$$

Since $F_0'''' \leqslant 0$, F_0, $F_0'' \geqslant 0$, the second integral in the numerator is negative and if eigenvalues exist, they must be positive.

When the differential equation for F_n possesses an eigenfunction and a forcing term, which we have denoted by $G_n(\eta)$, a solution will exist only if $G_n(\eta)$ satisfies an integral relation.‡ To derive this result let z_0 denote the eigenfunction corresponding to $\lambda = \lambda_0$ and put $F_n = F_0' w_n$ with $z_n = w_n'$; thus

$$(F_0'^3 e^{-q} z_0')' + F_0'^2 e^{-q} P(\eta, \lambda_0) z_0 = 0, \tag{4.28}$$

and

$$(F_0'^3 e^{-q} z_n')' + F_0'^2 e^{-q} P(\eta, \lambda_0) z_n = F_0'^2 e^{-q} G_n. \tag{4.29}$$

Multiply (4.29) by z_0 and (4.28) by z_n, subtract the equations and integrate by parts over the infinite range. Using the boundary conditions we find

$$\int_0^\infty F_0'^2 e^{-q} z_0(\eta) G_n(\eta)\,d\eta = 0.\S \tag{4.30}$$

Unless fortuitous circumstances prevail, this equation will not be satisfied and a modification of the previous terms in the asymptotic expansion must be made to avoid this contradiction. This procedure has been called 'switchback' and the

† The convergence of the integrals at the upper limit is ensured by the factor $e^{-q} \sim \exp(-\frac{15}{32}A_0\eta^{\frac{8}{3}})$.

‡ Without a forcing term, the solution for F_n is an arbitrary multiple of the eigenfunction.

§ This equation is the well-known requirement that the forcing term must be orthogonal to the eigenfunction to avoid resonance.

remedy involves the inclusion of the term $(\xi^{\frac{4}{3}n}\ln\xi)f_n(\eta)$ in (4.13) for the following reasons: (1) f_n will satisfy the homogeneous equation of F_n and will therefore be an arbitrary multiple of the eigenfunction, (2) the inclusion of f_n will introduce a new term $cg_n(\eta)$, c being an arbitrary constant, into the forcing term for F_n. Thus the integral relation (4.30) will be replaced by

$$\int_0^\infty F_0'^2 e^{-q} z_0(\eta)\,[G_n(\eta)+cg(\eta)]\,d\eta = 0, \tag{4.31}$$

and a value for c may be chosen so that (4.31) is satisfied. In rare cases this procedure will not be sufficient because the additional forcing terms, depending on f_n, will cancel in the equation for F_n and we must consider 'double switchback', i.e. the introduction of a logarithmic term at an earlier stage in the asymptotic expansion.

Although the evolution of the structure of these asymptotic expansions is interesting, the most important point is that the multiples of the eigenfunctions are not known and depend on the initial conditions for the boundary-layer motion upstream. This is not surprising physically, yet for the free streamline problem considered here, the singularity in the pressure gradient is strong enough to completely determine the skin friction near the free streamline to first order, independently of the initial conditions. The initial conditions will enter at the second order via an eigenfunction.

(*d*) *Eigenvalues and the skin friction*

A numerical computation, described in the appendix, yielded the first two eigenvalues

$$\lambda_0 = 1{\cdot}3157\ldots,\quad \lambda_1 = 3{\cdot}2562\ldots.$$

The corresponding values for $4n$ are $1{\cdot}2629\ldots$ and $9{\cdot}0249\ldots$. Since these values are not integers, there is no need to introduce logarithms thus far. When the eigenfunctions are included, the first few terms in the expansion for $F(\xi,\eta)$ [see equation (4.13)] are

$$F(\xi,\eta) \sim F_0(\eta)+t^{\gamma}F_{\gamma}(\eta)+t^{2\gamma}F_{2\gamma}(\eta)+t^{3\gamma}F_{3\gamma}(\eta)+tF_1(\eta)+o(t), \tag{4.32}$$

where $\gamma = 0{\cdot}3157\ldots$, F_γ is an eigenfunction and $F_{2\gamma}$ and $F_{3\gamma}$ its offspring.

The skin friction τ_w is given by

$$\tau_w = (\partial u/\partial Y)_{Y=0} = 2^{\frac{3}{2}}k^3t^{-\frac{1}{4}}[F_0''(0)+c't^{\gamma}+c''t^{2\gamma}+c'''t^{3\gamma}+O(t)], \tag{4.33}$$

where c', c'', c''' are unknown and dependent on the upstream boundary-layer flow. Thus,

$$\tau_w = 2^{\frac{5}{4}}k^{\frac{11}{4}}(3{\cdot}014015\ldots)(-x)^{-\frac{1}{8}}+\ldots, \tag{4.34}$$

and to first order the skin friction is independent of the boundary-layer motion upstream.

5. Principal asymptotic expansion

The boundary condition (4.5) has not been imposed thus far because the solution found in §4 is not uniformly valid for large Y. To satisfy

(4.5) an 'outer' asymptotic expansion† based on (t, Y) is assumed to be of the form

$$\Psi^o \sim \Psi_0(Y)+t^{\frac{1}{2}}\Psi_1(Y)+t\Psi_2(Y)+o(t) \sim \sum_{n=0} t^n\Psi_n(Y) \quad (Y > 0, t \geqslant 0), \tag{5.1}$$

where
$$\Psi_0(Y) = \int_0^Y U_s(Y)\,dY. \tag{5.2}$$

Substituting (5.1) into (4.3) and equating to zero the coefficients of each power of t, we find

$$\Psi_0'\Psi_1' - \Psi_0''\Psi_1 = 0, \tag{5.3}$$

$$\Psi_0'\Psi_2' - \Psi_0''\Psi_2 = -1, \tag{5.4}$$

$$\Psi_0'\Psi_n' - \Psi_0''\Psi_n = H_n(\Psi_0, \Psi_1 \ldots, \Psi_{n-1}), \tag{5.5}$$

where primes now denote differentiation with respect to Y. H_n depends on previous Ψ_m's $(m < n)$ (and may be zero) and n is not restricted to integer values. For $Y \to \infty$, (4.5) requires

$$\Psi_0'(Y) \sim 1, \quad \Psi_2'(Y) \sim -1 \quad \text{for} \quad Y \to \infty, \tag{5.6}$$

and
$$\Psi_n'(Y) \sim 0 \quad \text{for all other } n \text{ when} \quad Y \to \infty. \tag{5.7}$$

The solutions of (5.3)–(5.5) are

$$\Psi_1(Y) = k_0\Psi_0'(Y), \tag{5.8}$$

$$\Psi_2(Y) = k_1\Psi_0'(Y) - \Psi_0'(Y)\int^Y [\Psi_0'(s)]^{-2}ds, \tag{5.9}$$

$$\Psi_n(Y) = k_n\Psi_0'(Y) + \Psi_0'(Y)\int^Y H_n(s)\,[\Psi_0'(s)]^{-2}ds, \tag{5.10}$$

where k_n are constants of integration. If $\Psi_0''(Y) \sim 0$ with an exponentially small error, Ψ_2' will satisfy (5.6) and, by induction, Ψ_n' will satisfy (5.7); thus (4.5) will be satisfied by the outer expansion.

(a) *The matching requirement*

It remains to be shown that (5.1) provides an extension of the asymptotic expansion (4.8) to the region $Y = O(1)$. If we assume this is true (and verify it *a posteriori* by its consistency), the matching requires

$$\lim_{\eta\to\infty}\Psi^i(\xi, \eta) = \lim_{Y\to 0}\Psi^o(t, Y). \tag{5.11}$$

The limit on the left should be interpreted as one in which Y is non-zero, fixed but very small, with $\xi \to 0$. Once the asymptotic form of $\Psi^i(\xi, \eta)$ for $\eta \to \infty$ is obtained,‡ the result should be expressed in terms of t, Y and should agree, term-by-term, with the right-hand side of (5.11). The condition (A) requiring that $F_n(\eta)$ be, at

† This type of expansion, first used by Goldstein (1930, see §2.4), has been called the principal asymptotic expansion by Kaplun (1967).

‡ Neglecting exponentially small terms, if there are any.

most, algebraically large† when $\eta \to \infty$ can now be justified as follows: it is expected that the terminal velocity profile determined from $\Psi_0(Y)$ will satisfy a no-slip condition at $Y = 0$. If $\Psi_0 = O(Y^{\alpha})$ $(\alpha > 1)$ for $Y \to 0$, the complementary solution for $\Psi_n \propto \Psi_0' = O(Y^{\alpha-1})$. Using induction, it may be shown that all the particular solutions for Ψ_n will be algebraic functions of Y (and $\ln Y$) for $Y \to 0$. Thus, (5.11) could not be satisfied if Ψ^i contained exponentially large terms for $\eta \to \infty$.

(b) *The terminal velocity profile*

The velocity profile $U_s(Y) = \Psi_0'(Y)$ is determined by the first term of the asymptotic expansion of each $F_n(\eta)$ for $\eta \to \infty$ [see (5.1) and (5.11)]. Thus using (4.8), (4.13) and writing $F_m(\eta) \sim A_0^m \eta^{\frac{1}{3}(4m+5)}$‡

$$\Psi_0(Y) = 2^{\frac{1}{2}}k(2^{\frac{1}{2}}kY)^{\frac{5}{3}} \sum_{n=0} A_0^n (2^{\frac{1}{2}}kY)^{\frac{4}{3}n}, \tag{5.12}$$

where the sum extends over all n, including the eigenvalues and the family of terms they beget. The first five terms involve the following powers of Y:

$$Y^{\frac{5}{3}},\ Y^{2\cdot 087\ldots},\ Y^{2\cdot 508\ldots},\ Y^{2\cdot 929\ldots},\ Y^{3}.$$

The forcing terms, due to $(U_e')^{-1}$, first enter the expansion with the term Y^3.

6. Comparison with numerical data

Some numerical integrations, using an explicit finite difference method, were carried out by Ackerberg and Phillips (private communication) in the hope that the form of the skin friction singularity near the free streamline might be deduced. The method of integration, which employed Mises variables (t, Ψ), has been described in Ackerberg (1968)§ and the details will not be repeated here. Accurate results very near the free streamline could not be obtained for the following reasons: (1) The method for finding the skin friction involves the fitting of the series expansion for $u(t, \Psi)$, which proceeds in powers of $\Psi^{\frac{1}{2}}$, to the numerical results at each step (see equation (A2.12) of Ackerberg 1968). When $t \to 0+$, the coefficients of this series become singular due to the pressure gradient singularity, and truncating the series at any stage introduces spurious singularities which are not related to the one being sought. (2) The terminal profile (5.12) indicates that for $\Psi \to 0$, the first term of $u(0, \Psi) \propto \Psi^{\frac{2}{5}}$ and not $\Psi^{\frac{1}{2}}$, which is the case for any $t > 0$. The transition from one form to the other as $t \to 0+$ must be intimately related to the infinite number of near singular terms in the series expansion for $t > 0$, and it would have been remarkable to have obtained accurate results.|| In spite of these difficulties some numerical results for the boundary-layer flow on a

† Terms of the form $(\ln \eta)^k \eta^m$ (which we consider algebraic) might also appear.

‡ Referring to equation (4.18) $A_n^0 \equiv A_n$.

§ The points at the edge of the boundary layer now require a different treatment.

|| The most serious objection to using Mises variables for studying boundary-layer separation in adverse pressure gradients is that for $\Psi \to 0$, $u(x_s, \Psi) \propto \Psi^{\frac{1}{3}}$ (x_s being the separation point), whereas for $x < x_s$, $u(x, \Psi) \propto \Psi^{\frac{1}{2}}$. In the past most authors have objected to the singularity in $\partial u/\partial \Psi$ for $x < x_s$ at $\Psi = 0$, which may be handled with ease, and have overlooked the nasty transition problem from $\Psi^{\frac{1}{2}}$ to $\Psi^{\frac{1}{3}}$.

finite flat plate set perpendicular to a uniform stream were obtained. The potential flow was assumed to be of the Kirchhoff–Rayleigh type with free streamlines attached at the salient edges ($\bar{x} = \pm l$), and the velocity, $U_e(x) = s$, is given implicitly by (see Ackerberg & Glatt 1968)

$$x = s(s^2+3)(1+s^2)^{-2}+\tfrac{1}{2}\sin^{-1}[2s(1+s^2)^{-1}] \quad (0 \leqslant s \leqslant 1). \tag{6.1}$$

Here the length scale $L = 4(\pi+4)^{-1}l$, l being the half-breadth of the plate, and the value of the inverse sine is in the range $(0, \frac{1}{2}\pi)$.

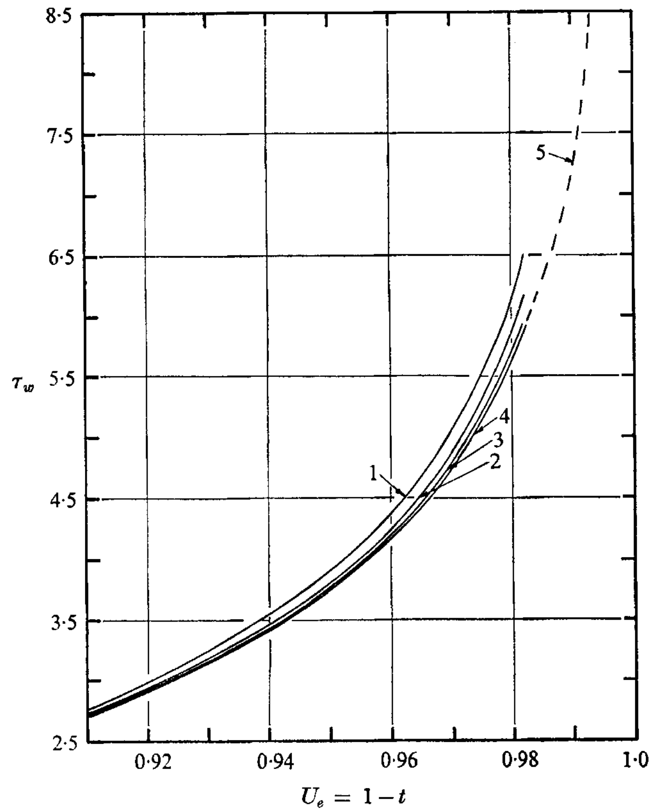

FIGURE 5. Comparison of the asymptotic theoretical result with finite difference calculations. Curve 5, theoretical result using (4.33) with $2^{\frac{1}{2}}k = 1$ [see (4.7) and (6.1)] and using a least squares fit with curve 4 to obtain $c' = -2{\cdot}818\ldots$, $c'' = -0{\cdot}961\ldots$. Numerical results: curve 4, $h = 0{\cdot}005$; curve 3, $h = 0{\cdot}010$; curve 2, $h = 0{\cdot}020$; curve 1, $h = 0{\cdot}040$.

The results for the skin friction, obtained for four different $\Delta\psi's(=h)$ [$h =$ 0·04, 0·02, 0·01, 0·005], are shown in figure 5. The dashed curve is the theoretical result from (4.33) with values of c' and c'' determined from fitting with the numerical data from curve 4 using the method of least squares. In this author's view, the agreement between the asymptotic expansion and the numerical data is quite good. A more detailed comparison using the velocity profiles will be made in a subsequent paper.

With numerical results for four different mesh sizes, it is tempting to try to improve the results by some extrapolation method. This does not seem possible, however, because an artificial singularity of the form $\tau_w \propto h^{\frac{1}{3}}(1-s)^{-\frac{2}{3}}$ develops due to the truncation of the series expansion for $u(t, \psi)$ when $\psi \to 0$, and it is not clear how to remove it; even if it is removed in some way, it is not obvious how to interpret the remainder.

7. Summary and discussion

It has been shown that for problems involving 'shearaway', with a pressure gradient of the form (3.10), the skin friction at the edge is singular and proportional to $(-x)^{-\frac{1}{8}}$, to first order. The constant of proportionality depends on the potential flow solution and is independent of the boundary-layer motion upstream. The upstream boundary-layer flow influences the skin friction at the second order through the inclusion of an eigenfunction. The terminal velocity profile $U_s(Y)$, for $Y \to 0$, contains non-integral and integral powers of Y, the first power being $Y^{\frac{4}{3}}$.

A comparison of theoretical results for the skin friction with some obtained by numerical integration showed good agreement. This author believes that the numerical integration of the boundary-layer equations for flows of this type is a challenging problem which deserves further consideration. A subsequent paper will compare the asymptotic theory with the numerical results in more detail and will extend the analysis to axisymmetric free streamline flows.

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Appendix. Eigenvalue calculation

Solutions of (4.24) subject to (4.25) and the condition following (4.25) are possible only for special values of λ. Near $\eta = 0$, two linearly independent solutions of (4.24) may be found which start with multiples of

$$1 \quad \text{or} \quad \eta^{-2}.$$

When $\eta \to \infty$, two linearly independent solutions commence with multiples of

$$\eta^{\frac{4}{3}(\lambda-1)} \quad \text{or} \quad \eta^{-\frac{1}{3}(4\lambda+1)} \exp\{\tfrac{15}{32}A_0\eta^{\frac{8}{3}}\}.$$

If we denote a solution which is finite at $\eta = 0$ by z_1, and a solution which grows algebraically for $\eta \to \infty$ by z_2, our task is to find those values of λ for which z_1 is a constant multiple of z_2.

A simple numerical method for finding these values is as follows: Let $\eta = r$ be a value in the range $0 < \eta < T$, where T is large enough so the WKBJ approximation can be used to approximate the solution z_2. Then, to a fair degree of accuracy, z_2' and z_2 are related by the equation obtained from (4.24), with the term z'' omitted. Choosing values for λ, $z_2(T)$, and $z_1(0)$, we may integrate backwards to find $z_2(r)$ and forwards to determine $z_1(r)$. At $\eta = r$, compute the Wronskian,

$$W_1(r;\lambda) = 1 - \frac{z_1(r)z_2'(r)}{z_1'(r)z_2(r)}, \tag{A 1}$$

which has been written in this way to avoid any problems with scaling. When W_1 vanishes, an eigenvalue has been found. By using Newton's method to correct successive values of λ, the eigenvalues may be found to 3 or 4 significant figures with 4 or 5 iterations. Some difficulty might be encountered if one of the terms in the denominator of (A 1) vanishes. Therefore it is also convenient to compute $W_2(r;\lambda)$ which is obtained from (A 1) by inverting the last term; thus

$$W_2 = W_1/(W_1 - 1),$$

and W_1 and W_2 vanish together. The eigenvalues of §4 were found by choosing $T = 20$, and $r = 1$. To compute larger eigenvalues, $z_2(T)$ must be chosen large enough to yield a value $z_2(r)$ which is not so small that it is inaccurate due to the numerical integration.

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